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WEIGHT FUNCTIONS ON THE KNESER GRAPH AND THE SOLUTION OF AN INTERSECTION PROBLEM OF SALI

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Let X, Y be finite sets and suppose that $\mathcal{F}$ is a collection of pairs of sets (F,G), $F\subset X$, $G\subset Y$ satisfying $|F\cap F'|\geq s$, $|G\cap G'|\geq t$ and $|F\cap F'|+|G\cap G'|\geq s+t+1$ for all (F,G), $(F',G')\in\mathcal{F}$. Extending a result of Sali, we determine the maximum of $\mathcal{F}$.

1. Introduction

Let X be a finite set and k be an integer. We denote by $\binom{X}{k}$ all k-element subsets of X. Let us construct the Kneser graph G on $\binom{X}{k}$ as follows. The vertex set of G is $\binom{X}{k}$ and two vertices are adjacent iff the corresponding two k-element sets are disjoint. Using a weight function on the Kneser graph, we prove some results on intersecting families. The main tool is the following.

Proposition 1. *Let $X=\{1,2,\dots,m\}$ and $G=(V,E)$ be the Kneser graph on $\binom{X}{k}$. Let further w_0 be a fixed constant. Let $w:V\to\mathbf{R}$ be a weight function with the following properties.*

(P1) *If $uv\in E$ and $w(u)=w_0$ then $w(v)\leq w_0$.*

(P2) *If $uv\in E$ and $w(u)=w_0+\binom{x}{n-l-1}$ for some x with $n-l-1\leq x\leq n-1$, then $w(v)\leq w_0-\binom{x}{l-1}$.*

Further, suppose that $n\geq 2l$ and $l/n\geq k/m$. Then $\sum_{v\in V}w(v)\leq |V|w_0$ holds.

As the first application of this proposition, we give a combinatorial proof of the following theorem, which is a special case of a result in [3].

Theorem 1. *Let X, Y be finite sets with $m=|X|\geq 2k$, $n=|Y|\geq 2l$. Suppose that $\mathcal{F}\subset\binom{X}{k}\times\binom{Y}{l}=\{(F,G):F\in\binom{X}{k},G\in\binom{Y}{l}\}$ is an intersecting family on $\binom{X\cup Y}{k+l}$.*

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Then it follows that

$$\frac{|\mathcal{F}|}{\binom{m}{k}\binom{n}{l}} \le \max\left\{\frac{k}{m}, \frac{l}{n}\right\}.$$

Next we extend a result of Sali. To state his result, we need some definitions. Let X and Y be finite sets. A family $\mathcal{F} \subset 2^X \times 2^Y$ is called (s,t,u)-intersecting if for every (F,G), $(F',G') \in \mathcal{F}$, $|F \cap F'| \ge s$, $|G \cap G'| \ge t$ and $|F \cap F'| + |G \cap G'| \ge u$. We define an s-intersecting family $K(X,s)$ on an m-element set X as the following.

$$K(X,s) = \begin{cases} \bigcup_{i=k}^{m} \binom{X}{i} & \text{if } m+s=2k \\ \left\{\bigcup_{i=k+1}^{m} \binom{X}{i}\right\} \cup \binom{X-\{x\}}{k} & \text{if } m+s=2k+1 \text{ and } x \in X. \end{cases}$$

Let us define $K(m,s)$ as the maximum size of s-intersecting families on an m-element set. By the Katona Theorem, it follows that $K(m,s) = |K(X,s)|$. Sali [13] proved the following.

Theorem 2. *Let X, Y be finite sets with $|X| = m$, $|Y| = n$. Suppose that $\mathcal{F} \subset 2^X \times 2^Y$ is $(1,1,3)$-intersecting. Then the following hold.*

(1) *If m, n are even,*

$$|\mathcal{F}| \le \binom{m-1}{m/2} K(n,3) + K(m,2)K(n,1).$$

(2) *If m is odd and n is even,*

$$|\mathcal{F}| \le \binom{m}{(m+1)/2} K(n,2) + K(m,3)K(n,1).$$

(3) *If m, n are odd,*

$$|\mathcal{F}| \le \binom{m}{(m+1)/2} \{K(n,2) + \frac{1}{n+1}\binom{n-1}{(n-1)/2}\} + K(m,3)K(n,1).$$

The bounds are sharp in the first two cases. ∎

The bound is not sharp in the last case. We extend the above result and give the sharp bound.

Theorem 3. *Let X, Y be finite sets with $|X| = m$ and $|Y| = n$. Suppose that $\mathcal{F} \subset 2^X \times 2^Y$ is $(s,t,s+t+1)$-intersecting. Then the following hold.*

(1) *If $m+s$, $n+t$ are odd,*

$$|\mathcal{F}| \le \binom{m-1}{(m+s-1)/2} K(n,t+2) + K(m,s+1)K(n,t).$$

(2) *If $m+s$ is even and $n+t$ is odd,*

$$|\mathcal{F}| \le \binom{m}{(m+s)/2} K(n,t+1) + K(m,s+2)K(n,t).$$

(3) *If $m+s$, $n+t$ are even and $m/s \le n/t$,*

$$|\mathcal{F}| \le \binom{m}{(m+s)/2} K(n,t+1) + K(m,s+2)K(n,t).$$ ∎

Example 1. The upper bounds in Theorem 3 are best possible. One of the extremal configurations is the following.

(1) If $m+s$ and $n+t$ are odd, fix an element $x \in X$ and define

$$\mathcal{F} = \{\binom{X-\{x\}}{(m+s-1)/2} \times K(Y,t+2)\} \cup \{K(X,s+1) \times K(Y,t)\}.$$

(2) If $m+s$ is even define

$$\mathcal{F} = \{\binom{X}{(m+s)/2} \times K(Y,t+1)\} \cup \{K(X,s+2) \times K(Y,t)\}.$$ ∎

2. Tools of proofs

One of the most useful results in extremal set theory is the Kruskal–Katona Theorem. Here we need it in the following version (cf. [11]). For a family $\mathcal{F}$ and an integer $l \ge 0$ define $\sigma_l(\mathcal{F}) = \{G : |G| = l, \exists F \in \mathcal{F}, G \subset F\}$.

Theorem 5. (Kruskal–Katona Theorem [9,6]) *Suppose that Y is an n-element set, $n \ge 2l$ and $\mathcal{H} \subset \binom{Y}{n-l}$ is a family of $(n-l)$-sets. Suppose further that $|\mathcal{H}| = \binom{n-1}{n-l} + \binom{x}{n-l-1}$ for some real number x, $n-l-1 \le x \le n-1$. Then $|\sigma_l(\mathcal{H})| \ge \binom{n-1}{l} + \binom{x}{l-1}$ holds.* ∎

Suppose that $\mathcal{A} \subset \binom{Y}{l}$ and $\mathcal{B} \subset \binom{Y}{l}$ are cross-intersecting, that is $A \cap B \ne \emptyset$ holds for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then clearly $\mathcal{B} \subset \binom{Y}{l} - \sigma_l(\mathcal{A}^c)$ holds, where $\mathcal{A}^c := \{Y - A : A \in \mathcal{A}\}$. Thus, for fixed $|\mathcal{A}|$ we can give an upper bound of $|\mathcal{B}|$ using the above theorem and this idea will be used in the proof of Theorem 1.

Let Δ denote the symmetric difference, that is $F \Delta G = (F-G) \cup (G-F)$. For a family $\mathcal{F} \subset 2^X$ and a positive integer t define $\partial_t(\mathcal{F}) = \{G \subset X : \exists F \in \mathcal{F}, |F \Delta G| \le t\}$. Given $|\mathcal{F}|$, what is $\min |\partial_t(\mathcal{F})|$? This problem was solved by Harper [5]. We need the following version of his result. (This follows from Harper's theorem and Lovász version of the Kruskal–Katona Theorem. cf. [12], [1]:pp.128–129.)

Theorem 5. (Numerical Harper Theorem) *Suppose that $\mathscr{F} \subset 2^X$, $|\mathscr{F}| = \binom{m}{m} + \binom{m}{m-1} + \cdots + \binom{m}{a+1} + \binom{m-1}{a} + \binom{x}{a-1}$ where x is a real number, $a-1 \le x \le m-1$. Then for $1 \le t \le a$ one has $|\partial_t(\mathscr{F})| \ge \binom{m}{m} + \cdots + \binom{m}{a-t+1} + \binom{m-1}{a-t} + \binom{x}{a-t-1}$.* ∎

Suppose that $\mathscr{A} \subset 2^X$ and $\mathscr{B} \subset 2^X$ are cross t-intersecting, that is $|A \cap B| \ge t$ holds for every $A \in \mathscr{A}$ and $B \in \mathscr{B}$. Then it follows that $\mathscr{B} \subset 2^X - \partial_{t-1}(\mathscr{A}^c)$. Thus, for fixed $|\mathscr{A}|$ we can give an upper bound of $|\mathscr{B}|$ again and this will be used in the proof of Theorem 3.

Finally we use the following slight extension of a lemma of Sali [13].

Lemma 1. *Let $\mathscr{F} \subset 2^X$ be an s-intersecting family on the m-element set X, and ε be an integer with $0 \le \varepsilon \le s$. Suppose that $F_1, F_2, \dots, F_h$ are the l-element sets in $\mathscr{F}$. Then there exist distinct sets $G_1, G_2, \dots, G_h \subset X$ such that $|G_j| = m - (l - s + \varepsilon)$ and $|F_j \cap G_j| = s - \varepsilon$ hold, $1 \le j \le h$.*

Proof. Let $\mathscr{F}_l = \{F_1, \dots, F_h\}$. In view of the Intersecting Kruskal–Katona Theorem [7],

$$|\sigma_{l-s+\varepsilon}(\mathscr{P})| \ge \left\{ \binom{2l-s}{l-s+\varepsilon} \Big/ \binom{2l-s}{l} \right\} |\mathscr{P}| \ge |\mathscr{P}|$$

holds for every $\mathscr{P} \subset \mathscr{F}_l$. This shows that $\mathscr{F}_l$ satisfies the Hall condition. So, there exist distinct sets $H_1, \dots, H_h$ satisfying $|H_j| = l - s + \varepsilon$, $H_j \subset F_j$, $1 \le j \le h$. Define $G_j := X - H_j$, then clearly $|F_j \cap G_j| = s - \varepsilon$, the result is proved. ∎

3. Proofs

Proof of Proposition 1.

Claim 1. *Suppose that $uv \in E$ and $w(u) \ge w(v)$. Then,*

$$kw(u) + (m-k)w(v) \le mw_0.$$

Proof. By (P1) this inequality clearly holds if $w(u) = w_0$. So suppose that $w(u) = w_0 + \binom{x}{n-l-1}$, $n-l-1 \le x \le n-1$. Then by (P2) we have $w(v) \le w_0 - \binom{x}{l-1}$. To prove our claim, we have to show that $k\binom{x}{n-l-1} \le (m-k)\binom{x}{l-1}$, or equivalently,

$$k(x-l+1)\cdots(x-n+l+2) \le (m-k)(n-l-1)\cdots l.$$

Since the LHS of the inequality is increasing with x, it suffices to show when $x = n-1$, that is, $k(n-l) \le (m-k)l$. This is equivalent to $l/n \ge k/m$, which completes the proof of Claim 1. ∎

Let H be an induced subgraph of G, where

$$V(H) = \{h_1 := \{1, 2, \dots, k\},\ h_2 := \{2, 3, \dots, k+1\},\ \dots,\ h_m := \{m, 1, 2 \dots, k-1\}\}.$$

Claim 2. $\sum_{h\in V(H)} w(h) \le |H|w_0$.

Proof. Consider $\max_{hh'\in E(H)}\{w(h)+w(h')\}$. By symmetry we may assume that this maximum is attained for the edge h_1h_t with $w(h_1)\ge w(h_t)$. Note that $h_1h_j \in E(H)$ for $k+1\le j\le m-k+1$ and $h_jh_{m-k+j}\in E(H)$ for $2\le j\le k$. Then we have

$$\begin{aligned}\sum_{h\in V(H)} w(h) &= \{w(h_1)+\sum_{j=k+1}^{m-k+1} w(h_j)\} + \sum_{j=2}^{k}\{w(h_j)+w(h_{m-k+j})\}\\ &\le w(h_1)+(m-2k+1)w(h_t)+(k-1)\{w(h_1)+w(h_t)\}\\ &= kw(h_1)+(m-k)w(h_t)\\ &\le mw_0. \quad \text{(by Claim 1)}\end{aligned}$$

This completes the proof of Claim 2. ∎

Since the automorphism group of the Kneser graph is transitive on its edges, by an averaging argument(cf. [8]) and Claim 2, we have

$$\sum_{v\in V} w(v) \le |V|w_0,$$

which completes the proof of Proposition 1. ∎

Proof of Theorem 1. We assume that $l/n \ge k/m$. Let $\mathcal{K} = (V,E)$ be the Kneser graph on $\binom{X}{k}$. We define a weight function $w: V \to \mathbf{N}$ by $w(v) := \#\{G\in\binom{Y}{l} : (v,G)\in\mathcal{F}\}$ for $v\in V$. Let $w_0 := \binom{n-1}{l-1}$. By the version of the Kruskal–Katona Theorem stated in the preceding section, w satisfies the properties (P1) and (P2) in Proposition 1. Hence we have

$$|\mathcal{F}| = \sum_{v\in V} w(v) \le |V|w_0 = \binom{m}{k}\binom{n-1}{l-1}.$$

∎

Proof of Theorem 3. Define $\mathcal{F}_X := \{F\subset X : \exists G\subset Y, (F,G)\in\mathcal{F}\}$ and $k_i := K(n,i)$. Consider a weight function $w:\mathcal{F}_X \to \mathbf{R}$ satisfying the following conditions.

(Q1) For all $F\in\mathcal{F}_X$, $w(F)\le k_t$.

(Q2) If $|F\cap H| = s$ for $F,H\in\mathcal{F}_X$ then $w(F)+w(H)\le k_t+k_{t+2}$.

Moreover, if $n+t=2b$ then we assume that w satisfies the following.

(Q3) If $|F\cap H|=s$ for $F,H\in\mathcal{F}_X$ and $w(F)=k_{t+1}$, then $w(H)\le k_{t+1}$.

(Q4) If $|F\cap H|=s$ for $F,H\in\mathcal{F}_X$ and $w(F)=k_{t+1}+\binom{x}{b-1}=k_{t+1}+\binom{x}{n-(b-t)-1}$ for $b-1\le x\le n-1$, then $w(H)\le k_{t+1}-\binom{x}{(b-t)-1}$.

Note that $\mathcal{F}_x$ satisfies (Q1)–(Q4) with the weight function $w(F) = \#\{G : (F,G)\in\mathcal{F}\}$, and we have $|\mathcal{F}| = \sum_{F\in\mathcal{F}_X} w(F)$. Indeed, (Q1) holds because of $|G_1\cap G_2|\ge t$ for all (F,G_1), $(F,G_2)\in\mathcal{F}$. (Q2) was proved by Sali [13], it can be proved also using the Numerical Harper Th., cf. [4]. (Q3) and (Q4) follow from the Numerical Harper Theorem applied to the families $\mathcal{A}=\{G:(F,G)\in\mathcal{F}\}$ and $\mathcal{B}=\{G:(H,G)\in\mathcal{F}\}$. Therefore, to conclude the proof it is sufficient to prove the following.

Proposition 2. *Let $\mathcal{F}_X \subset 2^X$ be an s-intersecting family and let $w:\mathcal{F}_X \to \mathbf{R}$ be a weight function satisfying* (Q1)–(Q4). *Then the following hold.*

(1) *If $m+s$, $n+t$ are odd,*

$$\sum_{F\in\mathcal{F}_X} w(F) \le \binom{m-1}{(m+s-1)/2} K(n,t+2) + K(m,s+1)K(n,t).$$

(2) *If $m+s$ is even and $n+t$ is odd,*

$$\sum_{F\in\mathcal{F}_X} w(F) \le \binom{m}{(m+s)/2} K(n,t+1) + K(m,s+2)K(n,t).$$

(3) *If $m+s$, $n+t$ are even and $m/s \le n/t$,*

$$\sum_{F\in\mathcal{F}_X} w(F) \le \binom{m}{(m+s)/2} K(n,t+1) + K(m,s+2)K(n,t).$$

Proof. Let $\mathcal{F}_l := \mathcal{F}_X \cap \binom{X}{l}$ and $h_l := |\mathcal{F}_l|$. We change $\mathcal{F}_X$ and w according to the following algorithm. Note that in this process, the total weight does not decrease and conditions (Q1)–(Q4) are satisfied.

Algorithm 1.

(i) *Define $l := \min\{i : h_i > 0\}$. If $l \le \lfloor (m+s-1)/2 \rfloor$ then go to* (ii), *otherwise end.*

(ii) *Let $h = h_l$ and $\mathcal{F}_l = \{F_1, \dots, F_h\}$. By Lemma 1, there exist $G_1, \dots, G_h \in \binom{X}{m-l+s}$ such that $|F_j \cap G_j| = s$ for $1 \le j \le h$. Define*

$$\mathcal{F}_X := \mathcal{F}_X \cup \{G_1, \dots, G_h\},$$
$$w(F_j) := k_{t+2} \quad \text{for} \quad 1 \le j \le h,$$
$$w(G_j) := k_t \quad \text{for} \quad 1 \le j \le h.$$

If $l \le \lfloor (m+s)/2 \rfloor - 1$ then go to (iii), *otherwise end.*

(iii) *Let $\mathcal{A} := \{F \in \binom{X}{m-(l-s+1)} : F \notin \mathcal{F}_X\}$. By Lemma 1, $|\mathcal{A}| \ge h_l$ holds. Define*

$$\mathcal{F}_X := (\mathcal{F}_X - \mathcal{F}_l) \cup \mathcal{A},$$
$$w(A) := k_{t+2} \quad \text{for} \quad A \in \mathcal{A}$$

and go to (i). ∎

After this process, we obtain that

$$\bigcup_{i=l}^{m} \binom{X}{i} \subset \mathcal{F}_X \subset K(X,s),$$

where $l=\lfloor (m+s)/2\rfloor+1$. If $m+s=2a+1$ then $w(F)=k_{t+2}$ for $F\in\mathcal{F}_a$. Moreover, note that $\mathcal{F}_a$ remains unchanged during this process in this case. Since it is s-intersecting, $\mathcal{F}_a^c=\{X-F : F\in\mathcal{F}_a\}$ must be intersecting. By the Erdős–Ko–Rado Theorem [2], $|\mathcal{F}_a|\le\binom{m-1}{m-a-1}=\binom{m-1}{a}$ follows.

Case 1. $m+s=2a+1$ and $n+t=2b+1$.

$$\sum_{F\in\mathcal{F}_X} w(F)\le\binom{m-1}{a}k_{t+2}+\sum_{j=a+1}^{m}\binom{m}{j}k_t$$
$$=\binom{m-1}{a}K(n,t+2)+K(m,s+1)K(n,t).$$

Case 2. $m+s=2a$ and $n+t=2b+1$.

$$\sum_{F\in\mathcal{F}_X} w(F)\le\sum_{F\in\mathcal{F}_a} w(F)+\sum_{j=a+1}^{m}\binom{m}{j}k_t$$
$$\le\binom{m}{a}\frac{K(n,t)+K(n,t+2)}{2}+K(m,s+2)K(n,t)$$
$$=\binom{m}{a}K(n,t+1)+K(m,s+2)K(n,t).$$

Case 3. $m+s=2a$, $n+t=2b$ and $m/s\le n/t$.
First, let us consider w on $V:=\binom{X}{a}$. Define $k:=a$, $w_0:=k_{t+1}$, and $l:=b-t$. Then (Q3) and (Q4) imply (P1) and (P2). Note that $m/s\le n/t$ implies $l/n\le k/m$. Applying Proposition 1, it follows that

$$\sum_{F\in V} w(F)\le\binom{m}{a}k_{t+1}.$$

Therefore we have

$$\sum_{F\in\mathcal{F}_X} w(F)\le\sum_{F\in\mathcal{F}_a} w(F)+\sum_{j=a+1}^{m}\binom{m}{j}k_t$$
$$\le\binom{m}{a}K(n,t+1)+K(m,s+2)K(n,t).$$

This completes the proof of Proposition 2 and so the proof of Theorem 3. ∎

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